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On Certain Ternary Cubic-Form Equations.

BY J. J. SYLVESTER.

CHAPTER I.

EXCURSUS B.—ON THE CHAIN RULE OF CUBIC RATIONAL DERIVATION.

I THINK it desirable, while the colors, so to say, are still wet on the palette, and my mind is still dwelling upon the subject which has been casually introduced in the note to the proem contained in the last number of the Journal (and there made use of to determine the number of in-and-exscribed k-laterals to a cubic), without waiting to put forth the titles which in natural order of sequence, perhaps, should immediately follow Title 1 of Section 2, to proceed at once to develop the theory of derivation which, irrespective of the casual use of it alluded to, will be found to be of essential importance when I reach that part of my proposed task which deals with soluble cubic-form equations, nor less so when, in Chapter II., I have to treat of insoluble cases of certain classes of cubic-form equations with four or more terms.

Title 1.—*On the Natural or Discontinuously Numbered Scale of Rational Derivatives to a Point on a Cubic Curve.*

Let us take any point on a cubic curve along with its successive tangentials *ad infinitum.* We may, by drawing straight lines through any two of these points, either contiguous or apart, to meet the curve, obtain an additional set of points, and thus form an enlarged system which may again be subjected to a like process of collineation or tangentialization, and such method of augmentation and amplification may be continued indefinitely. Every point thus obtained will obviously be a rational derivative of the original point (i. e. its co-ordinates will be rational integral functions of those of that point), and, at first sight, it would seem as if we might in this way obtain a network, or spread,* of rational derivatives; but I shall proceed to show that such is not the case, but that only a line or chain of points will be thus obtained, usually infinite in extent, although for certain positions of the initial point coming to a stop, and in other cases winding round and round upon itself so as still to include only a finite number of distinct points. It will be shown subsequently that, in order to complete the theory of the chain for the purposes of this memoir, it will be necessary to take into account the rational derivatives not merely from a single arbitrary point, but from such points, *combined with a point of inflexion*, and that this additional element will not alter the surprising fact of the absence of reticulation or spread, but merely bring about the insertion into the chain of points corresponding to missing numbers in it as first described, and to the duplication of the chain so completed, owing to every point in it having an opposite point also situated on the curve and collinear with it in respect to the given inflexion. This duplication will be of little importance in general to the arithmetical theory with which we shall be occupied, inasmuch as opposite points will correspond to the same arithmetical values, with merely a change of name between two out of the three variables which denote the co-ordinates of any point. First, let us consider the chain law of derivation when a point on the cubic curve alone is given. I shall call the original point 1, and its first and second tangentials 2 and 4 respectively, and in general use (m, n) to denote the point on a given cubic collinear with two points m, n also situated upon it.† Obviously, then, we shall have $1, 1 = 2 \quad 2, 2 = 4$, using $1, 1 \quad 2, 2$ to denote, in either case, two consecutive points upon the cubic. It is also obvious that if $m, n = p$ then $m, p = n$ and $n, p = m$, so that $1, 2 = 1 \quad 2, 4 = 2$.

Let us call $1, 4 = 5 \quad 2, 5 = 7 \quad 1, 7 = 8 \quad 2, 8 = 10 \quad 1, 10 = 11 \quad 2, 11 = 13$ and so on. It will be seen that no number which is a multiple of 3 is brought into existence by this process. Supposing a, b to be any two integers, neither of them divisible by 3, let us agree to signify by $a \ddagger b$ that of the two values $a + b, a - b$ which is not divisible by 3. The theorem to be established is that the point m, n collinear to m and n will have for its value $m \ddagger n$; as, for instance, $4, 4$, or the third tangential to 1, will have for its value 8, i. e. will be identical with $1, 7$, that is to say, with $1, [2, (1, 4)]$, where 2 and 4 are the first and second tangentials to 1, which amounts to a rule for obtaining the third tangential, when a point on a cubic and its first and second tangentials are given, by collineation alone. The *theory of residuation,* in its simplest form (see Salmon's Higher Plane Curves, 3d ed., p. 134)* teaches us that the rule of the older chemistry known by the name of double decomposition, viz. that $(a, b), (c, d) = (a, c), (b, d)$ is applicable to the same symbols regarded as points on a cubic curve. This rule of double decomposition is all that is required to prove the theorem in question.

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* Spread, as a noun (scarcely to be found in the dictionaries), I employ in the sense in which it occurs in the phrase *spread of foliage*. On this continent the word *spread* is also used to denote a thick coverlet or padded woollen quilt, laid over the bedclothes in winter to keep out the cold; also on both continents as a familiar name for a college banquet.

† Sometimes, however, it will be found more convenient to use $P_1, P_2 \ldots . P_n$; $P'_1, P'_2, \ldots . P_n'$ in lieu of $1, 2, \ldots . n$; $1', 2', \ldots . n'$.

Thus, e.g., in order to prove that $1, 7 = 4, 4$, I write $1, 7 = (1, 2), (2, 5) = (2, 2), (1, 5) = 4, 4$. Q. E. D.

So, to prove in general that $r, s = r \ddagger s$ I proceed as follows:

1° Suppose $r = 3i + 1$; $s = 3j + 1$, where $j - i$ is positive. Then $r, s = (3i - 1, 2), (3j + 2, 1) = (3i - 1, 1), (3j + 2, 2) = 3i - 2, 3j + 4 = r - 3, s + 3$. Hence $r, s = r - 3i, s + 3i = 1, s + r - 1 = s + r$.

2° Suppose $r = 3i - 1$; $s = 3j - 1$. Then $r, s = (3i - 2, 1), (3j + 1, 2) = (3i - 2, 2), (3j + 1, 1) = 3i - 4, 3j + 2 = r - 3, s + 3$, as before. Hence $r, s = r - 3(i - 1), s + 3(i - 1) = 2, s + r - 2 = s + r$.

3° Suppose $r = 3i - 1$; $s = 3j + 1$. Then $r, s = (3i - 2, 1), (3j - 1, 2) = (3i - 2, 2), (3j - 1, 1) = 3i - 4, 3j - 2 = r - 3, s - 3$. Hence $r, s = r - 3i + 3, s - 3i + 3 = 2, s - r + 2 = s - r$.

4° Suppose $r = 3i + 1$; $s = 3j - 1$. Then $r, s = (3i - 1, 2), (3j - 2, 1) = (3i - 1, 1), (3j - 2, 2) = 3i - 2, 3j - 4 = r - 3, s - 3$. Hence $r, s = r - 3i, s - 3i = 1, s - r + 1 = s - r$.

Collecting the four cases, it will be seen that I have proved, for all values of the points r, s in the chain, that $r, s = r \ddagger s$. Q. E. D.

The points 2^j correspond to tangentials of the i^{th} order to the point 1. It is obvious from the above theorem that no process of continued collineation or tangentialization performed upon these points can lead to any points extraneous to the series of points $1, 2, 4, 5, 7, 8 \dots$ which form a simple chain extending in general to infinity. Moreover, as it follows from the theory of residuation that any single point reached through the intervention of curves drawn through any number of points on a cubic can be reached by simple linear constructions, it follows that by no conceivable geometrical process can any rational point be reached not included in the numbered chain, and the inference becomes in the highest degree probable, and, as a matter of fact, is undoubtedly true (although the reasoning upon which it is here made to rest is not absolutely conclusive), that no rational deducts from a *general* point on a *general* cubic exist save those that belong to the numbered chain, the points upon which constitute what may properly be termed a self-contained *group*, infinite or finite (as the case may be) in regard to the number of terms which it contains. I shall presently determine the order of each successive derivative, meaning thereby the order in the co-ordinates of the initial point of any one of the three functions which express the co-ordinates of the derived one.*

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* The theory of residuation was originally brought by me before the Mathematical Society of London, and subsequently, in the form of questions, in the "Educational Times." Dr. Salmon makes no allusion to the fact of my applying the theory to curves of all orders: in the case of the quartic, the residual becomes a system of three points; of a quintic, a system of six points, and so on. I understood Professor H. S. Smith to say that he made use of my theory for the quartic in his memoir which gained half the prize for the subject set by the Academy of Sciences of Berlin, but which I have never seen.

The case in which the chain forms a closed polygon, which can only happen when for some number i the i^{th} tangential coincides with the initial point, has already been discussed in the note to the proem.

If the chain is an open but finite one, it is necessary that a tangential of some order shall fall upon a point of inflexion, in which case the succeeding tangentials remain fixed at that point, but otherwise continual new tangentials could be drawn. These are obviously necessary conditions of the chain being finite, whether it be an open chain or winding round upon itself; it remains to show that they are sufficient as well as necessary, but that will best appear after the theory of derivation from a general point combined with a point of inflexion has been discussed.

I shall begin with finding the co-ordinates X, Y, Z of a point on the cubic curve collinear with any two given points x, y, z; ξ, η, ζ. Let $X = \lambda x + \mu\xi$, $Y = \lambda y + \mu\eta$, $Z = \lambda z + \mu\zeta$; then $F(X, Y, Z) = \lambda^3 F(x, y, z) + \lambda^2\mu\left(\xi\frac{d}{dx} + \eta\frac{d}{dy} + \zeta\frac{d}{dz}\right)F(x, y, z) + \mu^3 F(\xi, \eta, \zeta) + \lambda\mu^2\left(x\frac{d}{d\xi} + y\frac{d}{d\eta} + z\frac{d}{d\zeta}\right)F(\xi, \eta, \zeta)$. Hence X, Y, Z will be the collineal to (x, y, z), (ξ, η, ζ) if

$$\lambda : \bar{\mu} :: \left(x\frac{d}{d\xi} + y\frac{d}{d\eta} + z\frac{d}{d\zeta}\right)F(\xi, \eta, \zeta) : \left(\xi\frac{d}{dx} + \eta\frac{d}{dy} + \zeta\frac{d}{dz}\right)F(x, y, z).$$

If now we write $F(x, y, z)$ under its canonical form $x^3 + y^3 + z^3 + Kxyz$, it will be found, on substituting for λ and $\bar{\mu}$ the quantities to which they are proportional, that

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* There is a further question, but which, as not material to the object of this memoir, I shall not discuss here, viz. the *degree* in the coefficients of each such derivative. For the tangential, the degorder (being that of the minor determinants of the matrix made up of the differential derivatives of the function and its Hessian) we know to be 4, 4. If $x ;y, z$, be the original co-ordinates, and X, Y, Z, those of the tangential, we know that $F(X, Y, Z)$ being zero when $F(x, y, z)$ (the given cubic) is zero, must be divisible by $F(x, y, z)$. The quotient will be of the degorder 13, 12 — 1, 3, i. e. 12, 9, and is in fact the skew covariant of F.

$$\begin{aligned} X &= (y^2\eta\xi - y\eta^2x + z^2\zeta\xi - z\zeta^2x) + K\,(yz\xi^2 - \eta\zeta x^2) \\ Y &= (z^2\zeta\eta - z\zeta^2y + x^2\xi\eta - x\xi^2y) + K\,(zx\eta^2 - \zeta\xi y^2) \\ Z &= (x^2\xi\zeta - x\xi^2z + y^2\eta\zeta - y\eta^2z) + K\,(xy\zeta^2 - \xi\eta z^2). \end{aligned}$$

But these expressions admit of a surprising simplification, viz.: we may neglect the terms not containing K, for it will be found that the quantities affected with the coefficient K are to each other in the same ratios as the other three corresponding groups in the values of X, Y, Z. Thus, ex. gr.

$$\begin{aligned} &(yz\xi^2 - \eta\zeta x^2)\,(z^2\zeta\eta - z\zeta^2y + x^2\xi\eta - x\xi^2y) \\ &- (zx\eta^2 - \zeta\xi y^2)\,(y^2\eta\xi - y\eta^2x + z^2\zeta\xi - z\zeta^2x) \\ &= (\xi y - x\eta)\,\{\xi\eta\zeta\,(x^3 + y^3 + z^3) - xyz\,(\xi^3 + \eta^3 + \zeta^3)\} \end{aligned}$$

hence $\quad X : Y : Z :: yz\xi^2 - \eta\zeta x^2 : zx\eta^2 - \zeta\xi y^2 : xy\zeta^2 - \xi\eta z^2.$

We might, instead of these simple expressions, take for X, Y, Z the other three groups and (using $x_1y_1z_1$; $x_2y_2z_2$ instead of x, y, z; ξ, η, ζ and (pq) to denote the determinant $p_1q_2 - p_2q_1$) say that X, Y, Z are the minor determinants of

$$\begin{matrix} x_1 \cdot x_2 & y_1 \cdot y_2 & z_1 \cdot z_2 \\ (yz) & (zx) & (xy), \end{matrix}$$

and these are actually the expressions found by Cauchy, and given by him in his *Exercices de Mathématiques*, Paris, 1826, p. 256, l. 18–21, pp. 257–60. I take this reference from a loose page of an article by M. Lucas, but have not access either to that article or to Cauchy's.

It is remarkable that Cauchy should have given quadrinomial expressions for the collineal to two given points on a cubic curve, or their connective, as I shall hereafter term it, when, as shown above, binomial ones fulfil the same purpose. The correctness of these remarkable formulæ admits of easy verification, as follows:

For greater simplicity denote x^3, y^3, z^3, xyz by u, v, w, μ; and $\xi^3, \eta^3, \zeta^3, \xi\eta\zeta$ by u', v', w', μ' respectively. Then $\Sigma\,(yz\xi^2 - \eta\zeta x^2)^3 = \Sigma\,(vwu'^2 - v'w'u^2) - 3\,\mu\mu'\,\{(u' + v' + w')\,\mu - (u + v + w)\,\mu'\} = \Sigma\,(vwu'^2 - v'w'u^2)$.

Also $\quad K(yz\xi^2 - \eta\zeta x^2)\,(zx\eta^2 - \zeta\xi y^2)\,(xy\zeta^2 - \xi\eta z^2)$

$$\begin{aligned} &= -\,Kxyz\,(\xi^3\eta^3z^3 + \eta^3\zeta^3x^3 + \zeta^3\xi^3y^3) + K\xi\eta\zeta\,(x^3y^3\zeta^3 + y^3z^3\xi^3 + z^3x^3\eta^3) \\ &= (u + v + w)\,(uv'w' + vw'u' + wu'v') - (u' + v' + w')\,(u'vw + v'wu + w'uv) \\ &= \Sigma\,(u^2v'w' - u'^2vw). \end{aligned}$$

Hence, giving X, Y, Z the values indicated by the formula, we find

$$X^3 + Y^3 + Z^3 + KXYZ = 0,$$

which equation depends, as seen, and as we know *a priori* must be the case, on the pure algebraical fact that $X^3 + Y^3 + Z^3 + KXYZ$ is a syzygetic function of $x^3 + y^3 + z^3 + Kxyz$ and $\xi^3 + \eta^3 + \zeta^3 + K\xi\eta\zeta$, taking no account of the function $\xi\eta\zeta\,(x^3 + y^3 + z^3) - xyz\,(\xi^3 + \eta^3 + \zeta^3)$, as that is itself a syzygetic function of the two others. If we call the syzygetic multipliers of those two Φ and F respectively, it will at once be seen from what precedes that

$$\Phi = 3\,\xi^2\eta^2\zeta^2xyz - \xi^3\eta^3z^3 - \eta^3\zeta^3x^3 - \zeta^3\xi^3y^3$$
$$F = 3\,x^3y^3z^3\xi\eta\zeta - x^3y^3\zeta^3 - y^3z^3\xi^3 - z^3x^3\eta^3.^*$$

I now proceed to apply the foregoing results to the problem of determining the order in the co-ordinates of any derivative numbered j (where $j = 3\,i \pm 1$), which may be called its index, and shall prove that *the order of any derivative is the square of its index.*† It will also be shown that each of the derivatives above referred to will be of the form xU, yV, zW, where U, V, W are quantics in x^3, y^3, z^3 as variables, since these quantities satisfy the equation

$$(xU)^3 + (yV)^3 + (zW)^3 + KxyzUVW = 0,$$

where $$Kxyz = -x^3 - y^3 - z^3.$$

From this it follows that, calling x^3, y^3, z^3; a, b, c respectively, the scheme of derivatives contains the various solutions of the algebraico-diophantine equation $aU^3 + bV^3 + cW^3 - (a + b + c)\,UVW = 0$, and that, supposing the law of the squares to be demonstrated, U, V, W will be of the order $\frac{1}{3}\,\{(3\,i \pm 1)^2 - 1\}$, i. e. $3\,i^2 \pm 2\,i$ in a, b, c, where i is any integer. We thus see that the above equation admits of solutions in which U, V, W are of the orders 1, 5, 8, 16, 21, 33, 40 . . . . respectively. It will hereafter be shown, in like manner, that the missing derivatives, whose indices are multiples of 3 (belonging to the arbitrary point and point of inflexion combined), will satisfy the equation

$$U^3 + V^3 + abcW^3 - (a + b + c)\,UVW = 0,$$

where U, V, W will be necessarily of the orders $3\,i^2 \pm 2\,i$, $3\,i^2 \pm 2\,i$, $(i \pm 1)\,(3\,i \pm 1)$

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* Thus, $$F = (yz\xi + zx\eta + xy\zeta)\quad(yz\xi + \rho zx\eta + \rho^2xy\zeta)\quad(yz\xi + \rho^2zx\eta + \rho xy\zeta)$$
$$\Phi = (\eta\zeta x + \zeta\xi y + \xi\eta z)\quad(\eta\zeta x + \rho\zeta\xi y + \rho^2\xi\eta z)\quad(\eta\zeta x + \rho^2\zeta\xi y + \rho\xi\eta z),$$
and it is worthy of notice that we have incidentally solved with quantic values for F, Φ, U, V, W the simultaneous algebraico-diophantine equations
$$U^3 + V^3 + W^3 = (a^3 + b^3 + c^3)\,\Phi - (\alpha^3 + \beta^3 + \gamma^3)\,F$$
$$UVW = abc\Phi - \alpha\beta\gamma\,F.$$

† The proof here supplied is sufficiently exact to dispel any reasonable doubt as to the truth of the law; but an exact proof which does not assume but demonstrates the non-existence of latent common measures to the reduced values of the co-ordinates of the connective to any two derivatives will be furnished under Title 3. — one of the most surprising feats of demonstration which it has ever fallen to the author's lot to accomplish.

respectively, i, as before, representing any integer. Thus we see that, if $a + b + c = 0$, the equations

$$aU^3 + bV^3 + cW^3 = 0 \text{ and } U^3 + V^3 + abcW^3 = 0$$

will admit of an infinite number of solutions in integers, when a, b, c are integer. This fact, as regards the latter equation, has been already pointed out by M. Lucas in this Journal, and previously by the Abbé Pepin in his memoir in Liouville's Journal, 2d series, Tome XV.

Let us begin with applying the formulæ to obtaining the co-ordinates of the tangential.

Let
$$x^3 + y^3 + z^3 + 3\,kxyz = 0$$

be the equation to the cubic. If we take x, y, z; $x + \delta x, y + \delta y, z + \delta z$ two consecutive points, their connective will be the tangential.

Applying the formulæ just obtained, we shall obtain for its co-ordinates expressions each of the form $P\delta x + Q\delta y + R\delta z$ with only one relation between $\delta x, \delta y, \delta z$. Hence, if we write $\delta z = \lambda\delta x + \mu\delta y$ the resulting ratios must be independent of λ and μ. Consequently we may make $\delta z = 0$. The two connectives then become

$$\begin{gathered} x, y, z \\ x + \delta x, y + \delta y, z, \end{gathered}$$

and the co-ordinates of the tangential will therefore be proportional to

$$yz\,(x + \delta x)^2 - z\,(y + \delta y)\,x^2 : zx\,(y + \delta y)^2 - z\,(x + \delta x)\,y^2 : z^2\,(xy - (x + \delta x)\,(y + \delta y))$$

i. e. to
$$x\,(2\,y\delta x - x\delta y) : y\,(2\,x\delta y - y\delta x) : z\,(x\delta y + y\delta x)$$

where
$$\delta x : \delta y :: y^2 + kxz : x^2 + kyz.$$

Hence the co-ordinates required are as

$$x\{2\,y^3 + x^3 + 3\,kxyz\} : y\{-2\,x^3 - y^3 - 3\,kxyz\} : z\,(x^3 - y^3),$$

i. e. as
$$x\{y^3 - z^3\} : y\{z^3 - x^3\} : z\{x^3 - y^3\},$$

a result which appears to have been first found by Cauchy for the general form, but previously by Euler, and before him by Fermat, for the case $k = 0$.

If we write a, b, c, instead of x, y, z, and call the co-ordinates of the tangential x, y, z, we might find their values by virtue of the condition that the connective of a, b, c and x, y, z is a, b, c over again. This furnishes the equations

$$\begin{aligned} bcx^2 - a^2yz &= am \\ cay^2 - b^2zx &= bm \\ abz^2 - c^2xy &= cm, \end{aligned}$$

which may be satisfied by writing

$$x = a(b^3 - c^3)\rho \ ; \ y = b(c^3 - a^3)\rho \ ; \ z = c(a^3 - b^3)\rho \ ;$$
$$(a^6 + b^6 + c^6 - a^3b^3 - b^3c^3 - a^3c^3)\rho^2 = m \ ;$$

but whether or not the above is necessarily the only possible solution is not quite clear *a priori*, and *a posteriori* it looks as if the solutions might be manifold.

The co-ordinates of the point whose index is 4, i. e. of the second tangential, will be those of the first tangential to the point

$$x(y^3 - z^3) : y(z^3 - x^3) : z(x^3 - y^3), \text{ viz.}$$
$$x(y^3 - z^3)\{y^3(x^3 - z^3)^3 + z^3(x^3 - y^3)^3\} : y(z^3 - x^3)\{z^3(y^3 - x^3)^3 + x^3(y^3 - z^3)^3\}$$
$$: z(x^3 - y^3)\{x^3(z^3 - y^3)^3 + y^3(z^3 - x^3)^3\},$$

and are of the order 16.

To find the co-ordinates of the point whose index is 5, we may take the connective of the one last found, and of x, y, z, i. e. of 4 and 1. Let us call them xU, yV, zW, and, for greater simplicity, denote x^3, y^3, z^3, for u, v, w. Then, omitting the common factor xyz,

$$U = (v - w)^2\{v(u - w)^3 + w(u - v)^3\}^2$$
$$- (w - u)(u - v)\{w(v - u)^3 + u(v - w)^3\}\{u(w - v)^3 + v(w - u)^3\},$$

with similar quantities (*mut. mut.*) set against V and W.

These quantities will have the common measure $u^2 + v^2 + w^2 - uv - uw - vw$.

To prove this let either one of its factors as $u + \rho v + \rho^2 w = 0$.

Then $$v - u = \rho^2(w - v) \text{ and } u - w = \rho(w - v),$$

and the representative of U above written becomes

$$\{(v - w)^2 - (w - u)(u - v)\}(w - v)^8 = (v^2 + w^2 + u^2 - vw - uw - uv)(w - v)^8 = 0.$$

Hence the representative of U vanishes with, and therefore contains

$$u^2 + v^2 + w^2 - uv - uw - vw$$

as a factor, and the same must evidently be true for the representatives of V and W; hence, U, V, W, will be of the order $10 - 2$ or 8, in u, v, w, and the co-ordinates xU, yV, zW, of the order $3 . 8 + 1$, i. e. of the order 25 in xyz.

The preceding demonstration depends essentially on the fact that my simplified formulæ for the co-ordinates of the connective of two points on a cubic fail, that is to say, become illusory, for a particular relation between the two points, as is easily seen; for let x, y, z ; $x, \rho y, \rho^2 z$ be two points on a cubic, then the formulæ for X, Y, Z, the connective's co-ordinates, become

$$(\rho y \cdot \rho^2 z - yz)x^2 \ ; \quad (\rho^2 z \cdot x - xz\rho^2)y^2 \ ; \quad (x \cdot \rho y - xy\rho^4)z^2.$$

i. e. all vanish, whereas it may be remarked that the general expressions given at page 62,

$$X = (y^2\eta\xi - y\eta^2x + z^2\zeta\xi - z\zeta^2x) + K(yz\xi^2 - \eta\zeta x^2)$$
$$Y = (z^2\zeta\eta - z\zeta^2y + x^2\xi\eta - x\xi^2y) + K(zx\eta^2 - \zeta\xi y^2)$$
$$Z = (x^2\xi\zeta - x\xi^2z + y^2\eta\zeta - y\eta^2z) + K(xy\zeta^2 - \xi\eta z^2),$$

become the minors of

$$\begin{matrix} x^2 & \rho y^2 & \rho^2z^2 \\ (\rho^2-\rho)\,yz & (1-\rho^2)\,zx & (\rho-1)\,xy\,; \end{matrix}$$

i. e. $(\rho^2-\rho)\,x\,(y^3-z^3), \quad (\rho-1)\,y\,(z^3-x^3), \quad (1-\rho^2)\,z\,(x^3-y^3)$,

which are the same as

$$x\,(y^3-z^3), \quad \rho^2y\,(z^3-x^3), \quad \rho z\,(x^3-y^3),$$

and remain perfectly valid.

This law of the failing case enables me to prove very easily the *Law of Squares*, as follows:

Suppose it proved that for all indices inferior to $6\,i$ the order of the derivative is equal to the square of its index; then, to prove that the same law is true up to $6\,(i+1)$, it is only necessary to consider the cases of $6\,i+1$, $6\,i+5$, for, as regards the index $6\,i+2$ and $6\,i+4$, the derivatives may be regarded as the tangentials of the derivatives to indices $3\,i+1$ and $3\,i+2$, and will consequently be of the orders $4\,(3\,i+1)^2$ and $4\,(3\,i+2)^2$, i. e. $(6\,i+2)^2$ and $(6\,i+4)^2$ respectively.

Let us further suppose that for derivatives whose indices are inferior to $6\,i$ the co-ordinates are of the form xU, yV, zW; U, V, W being quantics in x^3, y^3, z^3; then, obviously, from the mode of forming the tangential, this will be true for derivatives whose indices are $6\,i+2, 6\,i+4$: for the tangential to xU, yV, zW is $xU\,(y^3V^3-z^3W^3), yV\,(z^3W^3-x^3U^3), zW\,(x^3U^3-y^3V^3)$.

Let us consider the point (1) whose co-ordinates x, y, z satisfy the equation

$$x^3+\rho y^3+\rho^2z^3=0.$$

For such a point $\quad y^3-z^3 : z^3-x^3 : x^3-y^3 :: 1 : \rho : \rho^2$,

and the point (2) becomes $x, \rho y, \rho^2z$. Consequently the point (4) becomes $x\,(y^3-z^3), \rho y\,(z^3-x^3), \rho^2z\,(x^3-y^3)$ the same as $x, \rho^2y, \rho z$; hence the point (5), the connective of (1, 4), becomes $x\,(y^3-z^3), \rho y\,(z^3-x^3), \rho^2z\,(x^3-y^3)$, the same as $x, \rho^2y, \rho z$, so that, denoting the derivatives by their indices,

$$5=4 \qquad 7=1, 8=1, 1=2 \qquad 10=2, 8=2, 1=1$$
$$11=4, 7=4, 2=2 \qquad 13=2, 11=2, 2=4, \text{ etc.}$$

We have, thus, for all values of the point i

$$9\,i \pm 1\,,2\,, \pm 4 = 1\,,2\,,4,$$

when 1 is the point for which $x^3 + \rho y^3 + \rho^2 z^3 = 0$.

Hence, if $p\,,p'$ be any two points for which $p - p' = 3$, then $p\,,p'$ will be respectively identical with some two out of the three points $1\,,2\,,4$. And it will at once be seen that the simplified formulæ for the connective of any two of these three points become illusory.

Now the point $6\,i + 1$ is the connective of $3\,i - 1$ and $3\,i + 2$, and the point $6\,i + 5$ is the connective of $3\,i + 1$ and $3\,i + 4$.

Hence, in each of these cases, the simplified formulæ become illusory, i. e. the expressions for each of the co-ordinates vanish when $x^6 + y^6 + z^6 - x^3y^3 - x^3z^3 - y^3z^3$ vanishes, and must therefore contain it as a common measure. Moreover, the simplified formulæ for the connective co-ordinates for the points $xU, y\,V, z\,W$; $xU', y\,V', z\,W'$ will contain $x^2yz\,, y^2zx\,, z^2xy$, and will therefore have the common measure xyz. Hence the values of the co-ordinates when freed from these common measures will be of the order in $x\,,y\,,z$, $2\,(3\,i - 1)^2 + 2\,(3\,i + 2)^2 - 9$ for the point $6\,i + 1$, and $2\,(3\,i + 1)^2 + 2\,(3\,i + 4)^2 - 9$ for the point $6\,i + 5$, i. e. $(6\,i + 1)^2$ and $(6\,i + 5)^2$ respectively, and will obviously continue to be quantics in x^3, y^3, z^3 multiplied by $x\,,y\,,z$ respectively. Hence the theorem being true for index inferior to 6 is true universally.

It will be observed that any co-ordinate X of the point k must contain the X co-ordinate of the point k' where k' is any factor of k; for if $k = \delta k'$ the point k may be obtained by forming the point δ to the point k', and it has been shown that the δ derivative to any point has co-ordinates which contain respectively those of the initial point. Consequently the X co-ordinate to any point k may be resolved into factors containing a primitive part of the order τk (the totient of k) in the variables, and a non-primitive part containing the primitive part of each power of a prime contained in k, and with the exception of the single factor x all the others will be quantics in x^3, y^3, z^3; and, of course, the same remark applies to the other two co-ordinates Y and Z. We might obtain the point $m \ddagger n$ as the connective of $m\,,n$. In that case the simplified formulæ would give expressions of the order $2\,(m^2 + n^2)$ in $x\,,y\,,z$; and as the actual order of the co-ordinates in those variables is $(m \ddagger n)^2$, it follows that when $m - n \equiv 0$, Mod. 3, there will be a common measure (a symmetrical function of $x\,,y\,,z$) of the order $(m - n)^2$, and when $m + n \equiv 0$, Mod. 3, of the order $(m + n)^2$ running through those expressions, and it might be desirable to ascertain its form; but without waiting to solve this problem,* which is irrelevant to the matter in hand, I shall proceed at once to consider the derivatives corresponding to indices which are multiples of the number 3, to obtain which it is only necessary, as will be seen immediately, to combine one given point of inflexion with one arbitrary point of the curve. But, before doing so, it may be well to notice, that while the preceding investigation serves to show that the abridged formulæ for the connective co-ordinates possess the common measure $xyz\,(x^6+y^6+z^6-x^3y^3-x^3z^3-y^3z^3)$, it does not demonstrate categorically that there is no other; or that some power of the second factor above written other than the first might not be a common measure. Consequently, what we have strictly proved, as will be evident on reviewing the argument, is that the order to a derivative of the index $3i \pm 1$ cannot *exceed* the square of that index; but before I come to an end of the discussion I trust to be able to establish with *Dirichletian* rigor that the order is actually equal to the square of the index.†

Title 2. — *On the Completed or Continuously Numbered Scale of Rational Derivatives to an Arbitrary Point on a Cubic, of which one Point of Inflexion is given.*

Let I be the given point of inflexion, and let any point (or system of points) and another point (or system of points respectively) collinear with the former in respect to I be called opposites. It is obvious that $I, I = I$, or that the inflexion is its own opposite. It will be convenient to denote the opposite to any point by the same index, but accented.

We have, then, obviously,

$$p', p = I; (p')' = p \text{ and } (p', q)' = I\,(p', q) = (I, I), (p', q) = (I, p'), (I, q) = p, q'.$$

Let $I', 2 = 3$; $I', 5 = 6$; and in general $I', 3i - 1 = 3i$. This is matter of definition. Let, now, the infinite system $1, 2, 3, 4, 5, 6, 7 \ldots\ldots$ and its opposite be regarded as a single group. I say, 1°, that this will be a closed group, in the sense that a straight line drawn through any two points (contiguous or apart) of this double chain will cut the cubic in a third point included in the group.

2° That the new points will be rational in respect to the co-ordinates of the initial point and the given point of inflexion, and, 3°, that the order in the variables for every point, without regard to its relation to the modulus 3, will be, as before, the square of its index.

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* It is completely solved in the corollary to Title 5.

† This anticipation (for it was only such when these words were written) will be found fully realized under Title 5.

I proceed to show that the connective of any two points in the double chain may be expressed as a single point therein. Several cases present themselves according to the form of each of the two connected points in respect to the modulus 3 except when the indices are congruent in respect to that modulus.

When the residues (r, r'), in respect to that modulus, are dissimilar, the result will in general be different according as one of them (as r) belongs to the higher or lower index.

In what follows it is to be understood that $i \overset{=}{<} j$.

Theorem 1. To prove that $3i+1, (3j+1)' = 3j - 3i$

and $3i+2, (3j+2)' = (3j-3i)'$.

[This will imply that $(3i+1)', 3j+1 = (3j-3i)'$ and $(3i+2)', 3j+2 = 3j-3i$].

$$3i+1, (3j+1)' = (3i-1, 2), [(3j-1)', 2'] = (2, 2'), [3i-1, (3j-1)']$$
$$= (3i-1)', 3j-1 = [(3i-2)', 1'], (3j-2, 1) = (1, 1'), [(3i-2)', 3j-2]$$
$$= 3i-2, (3j-2)'.$$

Hence, $3i+1, (3j+1)' = 1, (3j-3i+1)' = (1, 2), [(3j-3i-1)', 2']$
$$= (2, 2'), [1, (3j-3i-1)'] = 1', (3j-3i-1) = 3j-3i$$
and $3i-1, (3j-1)' = I, [3i-2, (3j-2)'] = (3j-3i)'$.

Theorem 2. To prove that $3i+1, (3j-1)' = (3i+3j)'$

and $3i-1, (3j+1)' = 3i+3j$.

[This will imply that $(3i+1)', 3j-1 = 3i+3j$

and $(3i-1)', 3j+1 = (3i+3j)'$].

$$3i+1, (3j-1)' = 3i-1, 2; \; (3j+1)', 2' = (3i-1)', 3j+1$$
$$= [(3i-2)', 1'] \; (3j+2, 1) = 3i-2, (3j+2)'.$$

Therefore, $3i+1, (3j-1)' = 1, (3j+3i-1)' = (3i+3j)'$

and $3i-1, (3j+1)' = I, [(3i-1)', 3j+1] = 3i+3j$.

Collecting the results of these two theorems, we see that

$$\left.\begin{array}{l} 3i \pm 1, (3j+1)' = 3j \mp 3i = (3i \mp 1)', 3j-1 \\ \text{and} \quad 3i \pm 1, (3j-1)' = (3j \pm 3i)' = (3i \mp 1)', 3j+1 \end{array}\right\}, \qquad \text{(A)}$$

so that, using $p \overset{\circ}{-} q$ (where neither p nor q contains 3), to denote that one of the two numbers $p+q, p \sim q$, which is divisible by 3, (p, q') is always either $p \overset{\circ}{-} q$ or $(p \overset{\circ}{-} q)'$. Also

$$3i+1, (3j)' = (3i-1, 2), [1, (3j-1)'] = (1, 2), [3i-1, (3j-1)']$$
$$= (3j-3i)', 1 = (1', 3j-3i+1), (2, 1) = [(1', 1), (3j-3i+1, 2)]$$
$$= (3j-3i-1)';$$

again $3\,i\,,(3j+1)'=(3\,i-1\,,1')\,,[(3j-1)',2']=1',(3j-3\,i)'$
$=(1',2')\,,[1\,,(3j-3\,i-1)']=(1\,,1')\,,[(3j-3\,i-1)',2']=3j+1-3\,i;$

and lastly $3\,i,(3\,i+1)'=(3\,i-1\,,1')\,,[(3\,i-1)',2']=I\,,1'=1.$

Hence, collecting the results, $3\,\iota\,,(3\,i+1)'=(3\,i+1)\sim 3\,\iota$, whatever the relation of magnitude may be between i and ι.
Similarly,

$$3\,i-1\,,(3j)'=(3\,i+1\,,2),[1\,,(3j-1)']=(1\,,2),[3\,i+1\,,(3j-1)']$$
$$=1\,,(3\,i+3j)'=(3\,i+3j-1)';$$

$$(3\,i)',3j-1=[(3\,i-1)',1],(3j+1\,,2)=1\,,(3\,i+3j)'=(3\,i+3j-1)';$$

and $(3\,\iota)',3\,i-1=[(3\,i-1)',1]\,,(3\,i+1\,,2)=1\,,(6\,i)'=(6\,i-1)'.$

Hence, collecting the results, $3\,i-1\,,(3\,\iota)'=(3\,i+3\,\iota-1)'$, and we have

$$\left.\begin{array}{l}3\,\iota,(3\,i+1)'=(3\,i+1)\sim 3\,\iota\,;(3\,\iota)',3\,i+1=[(3\,i+1)\sim 3\,\iota]';\\ 3\,\iota\,,(3\,i-1)'=3\,i-1+3\,\iota\,;(3\,\iota)',3\,i-1=(3\,i-1+3\,\iota)'.\end{array}\right\}\text{(B)}$$

Also,

$$\left.\begin{array}{l}3\,\iota\,,3\,i-1=(3\,\iota-1\,,1')\,,(3\,i-2\,,1)=(3\,\iota-1\,,3\,i-2)'=[(3\,i-1)\sim 3\,\iota]'\\ 3\,\iota\,,3\,i+1=(3\,\iota-1\,,1')\,,(3\,i+2\,,1)=(3\,\iota-1\,,3\,i+2)'=(3\,\iota+3\,i+1)'.\end{array}\right\}\text{(B}'\text{)}$$

It remains only to determine the connectives of $3\,i\,,3\,\iota$ and of $3\,i\,,(3j)'$ or $(3\,i)',3j$, which is easily done, for

$$3\,i\,,3\,\iota=(3\,i-1\,,1')\,,(3\,\iota-1\,,1')=(1',1'),(3\,i-1\,,3\,\iota-1)=2',3\,i+3\,\iota-2.$$

Hence (by A) $3\,i\,,3j=(3\,i+3j)'$ and consequently $(3\,i)',(3\,\iota)'=3\,i+3\,\iota$.

Again $3\,i\,,(3j)'=(3\,i-1\,,1')\,,[(3j-1)',1]=(1\,,1'),[3\,i-1\,,(3j-1)']=$ (by theorem A) $I\,,(3j-3\,i)'=3j-3\,i$. Hence also $3j\,,(3\,i)'=(3j-3\,i)'$.

These three results may be designated theorem C, and theorems $A\,,B\,,B',C$ collectively prove that the original scale $1,2,4,5,7,8\,.\,.\,.\,.\,,$ which formed a closed system (so to say "group"), remains still closed when we complete it by insertion of multiples of 3, provided that we join on to the completed system $1,2,3,4,5,6,7\,.\,.\,.\,.$ the opposite system $1',2',3',4',5',6',7'\,.\,.\,.\,.$

In every case it will be observed the connective of two indices (disregarding the accent) is either their sum or their difference.

The double scale may be formed by alternate addition of 1 and $1'$ in the manner following:

$$1\,,1=2\qquad 1',2=3\qquad 1\,,3=4'\qquad 1',4'=5'\qquad 1\,,5'=6'\qquad 1',6'=7$$
$$1\,,7=8\qquad 1',8=9\qquad 1\,,9=10'\qquad 1',10'=11'\qquad 1\,,11'=12'$$

.... which gives the numbers $1, 2, 3, 4', 5', 6', 7, 8, 9, 10', 11', 12'$, etc.; and, in like manner, by interchanging $1, 1'$, we may obtain $1', 2', 3', 4, 5, 6, 7', 8', 9', 10, 11, 12$, etc.

The new points $3, 6, 9$ .... ; $3', 6', 9'$.... belong to the natural scales $1, 2, 5$ .... ; $1', 2', 5'$.... collectively and not respectively; and the accented and unaccented multiples of 3 might have had their significations interchanged without any impropriety. It is now necessary to extend the law of the order in the variables to these inserted points, and to prove that for them, as for the points in the natural scale, the order of any point, in the variables of the initial point, is the square of its index.

If the cubic be thrown into the canonical form $x^3 + y^3 + z^3 + kxyz$, the point $x = 1, y = -1, z = 0$ may be taken to represent I, and if x, y, z be the initial point 1, the co-ordinates of $1'$ (the connective of 1 and I) become by the general formula yz, zx, z^2, or, more simply, y, x, z.

To find 3, then, we have to take the connective of y, x, z and $x(y^3 - z^3)$, $y(z^3 - x^3), z(x^3 - y^3)$; its co-ordinates, accordingly, by the general formula, are

$$\begin{gathered} yz\,(z^3 - x^3)\,(x^3 - y^3)\,y^2 - x^3z\,(y^3 - z^3)^2 \\ xz\,(x^3 - y^3)\,(y^3 - z^3)\,x^2 - y^3z\,(z^3 - x^3)^2 \\ xy\,(y^3 - z^3)\,(z^3 - x^3)\,z^2 - yxz^2\,(x^3 - y^3)^2\,; \end{gathered}$$

or, neglecting the common factor z, the co-ordinates of 3 are

$$\begin{gathered} y^3\,(x^3 - y^3)\,(x^3 - z^3) + x^3\,(y^3 - z^3)^2 \\ x^3\,(y^3 - x^3)\,(y^3 - z^3) + y^3\,(z^3 - x^3)^2 \end{gathered}$$

and

$$xyz\,(z^3 - x^3)\,(z^3 - y^3) + xyz\,(x^3 - y^3)^2;$$

or

$$\begin{gathered} y^3x^6 + z^3y^6 + x^3z^6 - 3\,x^3y^3z^3 \\ x^3y^6 + z^3x^6 + y^3z^6 - 3\,x^3y^3z^3 \end{gathered}$$

and

$$xyz\,(z^6 + y^6 + x^6 - x^3y^3 - z^3x^3 - y^3z^3).$$

In the particular case where $x^3 + y^3 + z^3 = 0$, these expressions (writing for greater brevity L, M, N for x^3, y^3, z^3) become

$$\begin{gathered} ML^2 - (L + M)\,M^2 + L\,(L + M)^2 + 3\,LM\,(L + M) \\ LM^2 - (L + M)\,L^2 + M\,(L + M)^2 + 3\,LM\,(L + M) \\ xyz\,[(L + M)^2 + L^2 + M^2 - LM + (L + M)^2] \end{gathered}$$

or

$$\begin{gathered} L^3 + 6\,L^2M + 3\,LM^2 - M^3 \\ M^3 + 6\,M^2L + 3\,ML^2 - L^3 \\ 3\,xyz\,(L^2 + LM + M^2)\,; \end{gathered}$$

which remain equally good, as co-ordinates of the point 3 to the initial point x, y, z, when the cubic is $x^3 + y^3 + Cz^3$, as is easily seen by writing $C^{\frac{1}{3}}z = \zeta$.

The point 3, it follows from what precedes, is of the order 9 in the variables x, y, z, and the same will be true for $3'$, which is obtained from 3 by the interchange of x and y; but in order that these points may be arithmetically as well as algebraically rational, it is of course necessary that the given cubic may admit of being expressed under the form $Ax^3 + Ay^3 + Cz^3 + Kxyz$, where A, C and K are integers.

Again, since $6 = 3', 3'$, 6 is the 2 of $3'$, and similarly $6'$ is the 2 of 3; since $9 = 3', 6'$ and $6'$ is the 2 of 3, 9 is the 3 of 3. So again, since $12 = 3', 9'$ and $9'$ is the 3 of $3'$, 12 is the $(1, 3)$ of $3'$, i. e. the $4'$ of $3'$ or 4 of 3; and similarly $12'$ is the 4 of $3'$. So again,

$$15 = (3', 12') = (1, 4) \text{ of } 3' = 5 \text{ of } 3', \text{ and } 15' = 5 \text{ of } 3$$

$$18 = (3', 15') = (1, 5') \text{ of } 3' = 6' \text{ of } 3' = 6 \text{ of } 3, \text{ and } 18' = 6 \text{ of } 3'$$

$$21 = (3', 18') = (1, 6) \text{ of } 3' = 7' \text{ of } 3' = 7 \text{ of } 3, \text{ and } 21' = 7 \text{ of } 3'$$

$$24 = (3', 21') = (1, 7) \text{ of } 3' = 8 \text{ of } 3', \text{ and } 24' = 8 \text{ of } 3\,;$$

$$27 = (3', 24') = (1, 8') \text{ of } 3' = 9' \text{ of } 3' = 9 \text{ of } 3 \ldots.$$

Hence, in general,

$$9i + 3 = (3i + 1) \text{ of } 3\,; \quad 9i + 6 = (3i + 2)' \text{ of } 3; \quad \text{and } 9i = 3i \text{ of } 3.$$

Consequently

$$3^q(3i + 1) = (3i + 1) \text{ of } 3 \text{ of } 3 \text{ of } 3 \ldots. (q \text{ times repeated}),$$

and
$$3^q(3i + 2) = (3i + 2)' \text{ of } 3 \text{ of } 3 \text{ of } 3 \ldots. (q \text{ times repeated}).$$

From this it follows, obviously, that $3^q(3i \pm 1)$ and $[3^q(3i \pm 1)]'$ are each of the order $[3^q(3i \pm 1)]^2$ in the variables, and thus the law of the squares extends to all points alike in the completed scale.

TITLE 3. — *On Compound Derivation.*

The object of what follows is to show that any derivative of a derivative has for its index (due regard being paid to the accents) the product of the numerical values of the indices of the operator and operand derivatives, that is to say, the $i^?$ of $j^? = ij^?$; the mark of interrogation denoting either a blank or an accent, as the case may be. Thus, while connection involves addition or subtraction, composition involves a process of multiplication.

1° Let us consider the i of j when neither i nor j contains 3. Then

$$3k + 1 \text{ of } j = (2 \text{ of } j), (3k - 1 \text{ of } j) \text{ and } 3k + 2 \text{ of } j = (1 \text{ of } j), (3k + 1 \text{ of } j).$$

Suppose the theorem proved up to $3k-1$. Then

$$3k+1 \text{ of } j = 2j, 3kj - j = (3k+1)j$$

$$3k+2 \text{ of } j = j, 3kj + j = (3k+2)j.$$

Hence it is true up to $3(k+1)-1$, and, being true when $k=1$ (since 1 of $j = j$ and 2 of $j = j, j = 2j$), it is true universally.

In like manner, since 1 of $j' = j'$ and 2 of $j' = j', j' = I, (j, j) = (2j)'$, it may be shown that i of $j' = (ij)'$. Moreover

$$1' \text{ of } j = j', \text{ and therefore } 2' \text{ of } j = (1' \text{ of } j), (1' \text{ of } j) = j', j' = 2j'$$

and

$$(3k+1)' \text{ of } j = (2' \text{ of } j), [(3k-1)' \text{ of } j]$$

$$(3k+2)' \text{ of } j = (1' \text{ of } j), [(3k+1)' \text{ of } j];$$

so that, if the equation i' of $j = (ij)'$ holds good up to $i = 3k-1$,

$$(3k+1)' \text{ of } j = [(3k+1)j]', \text{ and } (3k+2)' \text{ of } j = [(3k+2)j]';$$

so that the equation i' of $j = (ij)'$ will hold good up to $3(k+1)-1$, and, being true for $k=1$, is true universally.

In like manner, since $1'$ of $j' = j$, it will follow that i' of $j' = ij$.

It remains to obtain the corresponding equations when i, j are one or both of them multiples of 3.

Since 3 of $3^q = (3^q, 3^q), (3^q)' = (2.3^q)', (3^q)' = 3^{q+1}$,

$$9 \text{ of } 3^q = 3 \text{ of } 3 \text{ of } 3^q = 3 \text{ of } 3^{q+1} = 3^{q+2}$$

$$27 \text{ of } 3^q = 3 \text{ of } 9 \text{ of } 3^q = 3 \text{ of } 3^{q+2} = 3^{q+3}, \text{ and so on.}$$

Hence

$$3^p \text{ of } 3^q = 3^{p+q}.$$

Again,

$$\begin{aligned} 3 \text{ of } 3j+1 &= (3j+1, 3j+1), (3j+1)' \\ &= 6j+2, (3j+1)' = 9j+3 \text{ by A.} \end{aligned}$$

Hence 3^2 of $3j+1 = 3$ of $9j+3 = (18j+6)', (9j+3)' = 27j+9$ by C,

3^3 of $3j+1 = 3$ of $27j+9 = (54j+18)', (27j+9)' = 81j+27$ by C,

and so on. Hence 3^p of $3j+1 = 3^p(3j+1)$.

Again,

$$\begin{aligned} 3 \text{ of } 3j+2 &= (3j+2, 3j+2), (3j+2)' \\ &= 6j+4, (3j+2)' = (9j+6)' \text{ by A.} \end{aligned}$$

Hence 3^2 of $3j+2 = 3$ of $(9j+6)' = 18j+12, 9j+6 = (27j+18)'$ by C,

and so on. Hence 3^p of $3j+2 = [3^p(3j+2)]'$.

Again,

$$\begin{aligned} 3j+1 \text{ of } 3^p &= (2 \text{ of } 3^p), (3j-1 \text{ of } 3^p) = (3^p, 3^p), (3j-1 \text{ of } 3^p) \\ &= (2.3^p)', (3j-1 \text{ of } 3^p) \end{aligned}$$

and

$$3j-1 \text{ of } 3^p = (1 \text{ of } 3^p), (3j-2 \text{ of } 3^p).$$

Suppose it true that $3j-2$ of $3^p=(3j-2)\,3^p$ for a certain value of j.

Then $\quad 3j-1$ of $3^p=3^p,(3j-2)\,3^p=[(3j-1)\,3^p]'$

and $\quad 3j+1$ of $3^p=(2\,.\,3^p)',[(3j-1)\,3^p]'=(3j+1)\,3^p$.

But 1 of $3^p=1\,.\,3^p$; hence, for all values of j,

$$3j+1 \text{ of } 3^p=(3j+1)\,3^p=3^p \text{ of } 3j+1$$
$$3j-1 \text{ of } 3^p=[(3j-1)\,3^p]'=3^p \text{ of } 3j-1.$$

Hence, by the well-known method of successive transformation, we obtain the following results:

When neither m nor n contains 3, when both contain 3, and when one of them contains 3 and the other is of the form $3j+1$, we have

$$m \text{ of } n=n \text{ of } m=m' \text{ of } n'=n' \text{ of } m'=mn$$
$$m \text{ of } n'=n' \text{ of } m=m' \text{ of } n=n \text{ of } m'=(mn)'.$$

In the remaining case (viz. when of m and n, one contains 3 and the other is of the form $3j-1$), we have

$$m \text{ of } n=n \text{ of } m=m' \text{ of } n'=n' \text{ of } m'=(mn)'$$
$$m \text{ of } n'=n' \text{ of } m=m' \text{ of } n=n \text{ of } m'=mn.$$

This completes the algorithm of rational derivation.

TITLE IV.— *On Pertactile or Periodic Points on a Cubic Curve.*

A pertactile point, or point of pluperfect tactility, on a general cubic is a point at which the cubic admits of a higher order of contact with another curve than is in general possible. Thus the points of inflexion are pertactile points, because a tangent at one of them will meet the curve in three consecutive points. The same is the case with Plücker's twenty-seven points, because at each of them a conic of closest contact will pass through six consecutive points, the sixth point in which any conic passed through five consecutive points cuts the curve coinciding, in this case, with the point of contact. So, in general, a curve of the i^{th} order can only be made to pass through $3i-1$ consecutive points situated at P; but if the i^{th} derivative of P is a point of inflexion, then the $3i^{\text{th}}$ point common to all curves of the i^{th} order passing through $3i-1$ consecutive points at P will coincide with P, so that such curves will pass through $3i$ consecutive points, and P may accordingly be termed a point of pluperfect tactility, or more briefly, a pertactile point.

To prove that this is the case, it is necessary, in the first place, to prove that, at a general point P in the cubic, the $3i^{\text{th}}$ point in which all curves of the i^{th} order passing through $3i-1$ consecutive points at P intersect the cubic, is the $(3i-1)^{\text{th}}$ derivative of P, which may be done inductively as follows:

Suppose P_{3i-1} is the residual of $3i - 1$ consecutive points at P. To find the residual of $3i + 2$ consecutive points there, we may combine $3i - 1$ giving the residual P_{3i-1}, two more of them giving the residual P_2, and one giving Q, R, any two points collinear with P. We then combine $(P_{3i-1}, P_2), (Q, R)$ and obtain P_{3i+1}, P_1 which gives P_{3i+2} as the required residual. Hence the theorem, being true for P_2 (the residual of two consecutive points at P) and true for $P_{3(i+1)-1}$ if true for P_{3i-1}, is true universally.

If, now, the residual of $3i - 1$ points at P is to fall at P we must have $P_1 = P_{3i-1}$.

1° Suppose $i = 3k - 1$, then $P_1, P_{i-1} = P_{i-1}, P_{3i-1}$, i. e. $P_i = P_{2i}$.

Hence P_i is a point of inflexion I, or, as we may express it, P is an i^{th} sub-derivative of such point, or $P = I_{\frac{1}{i}}$.

2° Suppose $i = 3k + 1$, then $P_1, P_2 = P_2, P_{3i-1}$, i. e. $P_1 = P_{3i+1}$.

Hence $P_1, P_{i+1} = P_{i+1}, P_{3i+1}$, i. e. $P_i = P_{2i}$, and, as before, $P = I_{\frac{1}{i}}$.

3° Suppose $i = 3k$.

Then $1, (i-1)' = (i-1)', 3i - 1$, i. e. $i' = 2i = i', i'$. Consequently i', and therefore also i, is a point of inflexion.

Hence, as in the other two cases, P is an i^{th} sub-derivative of a point of inflexion,* which may either be the point used to form the scale, or any of the eight other inflexions.†

It may be well to notice here that whilst P_i, when i does not contain 3, is, as already shown, of the form xU, yV, zW, it follows from the law of compound derivation, since P_3 is of the form $R, S, xyz\Theta$ (where R, S, Θ, like U, V, W, are quantics in x^3, y^3, z^3) that P_i, when i is a multiple of 3 or any power of 3, will be of the form $M, N, xyz\Omega$ (where M, N, Ω are still quantics in x^3, y^3, z^3).

Calling X, Y, Z any i^{th} derivative to $x^3 + y^3 + z^3 + kxyz = 0$, we must have $X^3 + Y^3 + Z^3 + kXYZ = 0$; and, in order for such derivative to be a point of inflexion, it is necessary and sufficient that $X = 0$ or $Y = 0$ or $Z = 0$; combining these equations respectively with the given cubic, we shall obtain, in all, 3 times $3i^2$ or $9i^2$ points, sub-derivatives of the i^{th} grade to one or other of the inflexions; but out of these, whether i be or be not divisible by 3, nine will correspond to $x = 0$, $y = 0$, or $z = 0$ combined with the curve, i. e. will be the points of inflexion themselves. Moreover, unless i be a prime number, it follows from the law of compound derivation, combined with the fact that x, y, z enter distributively or collectively into the derived co-ordinates X, Y, Z, that, if i' be any factor of i, and X', Y', Z' the co-ordinates of the i'^{th} derivative, Z will contain Z' and X, Y or Y, X, will contain X', Y' respectively. There will thus be a *primitive* part to X, Y, Z which results from driving out all the factors corresponding to any factor of i (unity included), and, if we suppose $i = a^{\alpha}.\, b^{\beta}.\, c^{\gamma}. \ldots$, the order of this primitive part in the variables x, y, z, it is easy to see, will be $a^{2(\alpha-1)}.\, b^{2(\beta-1)}.\, c^{2(\gamma-1)}. \ldots \{(a^2-1)\cdot(b^2-1)\,(c^2-1)\ldots\}$, which may be called the quadri-totient to i, and is the product of two factors, one the totient of i and the other what that totient becomes when $+1$ is substituted throughout for -1 in its expression, and which, if a name were needed for it, might be called the contra-totient.

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* A sub-derivative of an inflexion may conveniently be termed a sub-inflexion.

† The above formulæ show that $i, i' = 3i = 3i'$; hence $3i$ and $3i'$ coincide with the original point of inflexion, whereas $i, i', 2i, 2i'$ need not coincide with the original point of inflexion.

The number of proper, or primitive, i^{th} sub-derivatives of any point of inflexion will thus be the quadri-totient of i (just as the number of primitive i^{th} roots of unity is the totient), and the total number of pertactile points of the i^{th} grade, 9 times the quadri-totient of i.

It is easy to see that the points corresponding to the non-primitive factors of X, Y, Z satisfy, but in an *improper* manner, the conditions of the question. For, if i' is any sub-multiple of i (say $i' = \frac{i}{\delta}$) and P' is an i'^{th} subderivative of a point of inflexion, through P' may be drawn δ curves each of the order i' (constituting an improper curve of the order i), each passing through $3\,i'$ consecutive points, and consequently their *ensemble* passes through $\delta\,.\,3\,i'$ or $3\,i$ consecutive points. We have now obtained the generalization of the theorem of which the enumeration of the points of inflexion and Plücker's points constitute the two first steps, and it is very easy to calculate the number of pertactile points N of any given grade i. Thus for $i = 1, 2, 3,\ 4\ ,\ 5\ ,\ 6\ ,\ 7\ ,\ 8\ ,\ 9\ , 10,\ 11\ , 12 \ldots$

$$\frac{N}{9} = 1, 3, 8, 12, 24, 24, 48, 48, 72, 72, 120, 96 \ldots$$

The calculation is facilitated by the remark that if i, j are prime to each other, the number of $(ij)^{\text{th}}$ subderivatives to any one point of inflexion is the product of the number of i^{th} by the number of j^{th} subderivatives; the quadri-totient obeying the same law as the totient in this particular.

If i is the grade of the pertactile point P, so that $P_1 = P_{3i-1}$, then P_i is an inflexion, and P_{3i} is I, the original inflexion. Moreover

$$P_1 = P_1, P_2 = P_{3i-1}, P_2 = P_{3i+1}$$

$$P_2 = P_1, P_1 = P_1, P_{3i-1} = P_{3i-2} \text{ and also} = P_2, P_4 = P_{3i-2}, P_4 = P_{3i+2}$$

$$P_4 = P_2, P_2 = P_2, P_{3i-2} = P_{3i-4} \text{ and also} = P_2, P_{3i+2} = P_{3i+4}, \text{ and so on.}$$

And again, $P'_3 = P_3 I = P_3, P_{3i} = P'_{3i+3}$, and therefore $P_3 = P_{3i+3}$;

and $\quad P_{3i-3} = P'_{3i+3}, P_6 = P'_3, P_6 = P'_3$ whence $P_3 = P'_{3i-1}$;

$P_6 = P'_3, P'_3 = P'_{3i+3}, P'_{3i+3} = P_{6i+6}$ and also $= P_{3i-3}, P_{3i-3} = P'_{6i-6}$.

Thus in general, $P_{3r+1} = P_{3i\pm(3r+1)}$; $P_{3r-1} = P_{3r\pm(3r-1)}$

and $\quad P_{3r} = P_{3i+3r} = P'_{3i-3r}$.

Thus the natural scale $\quad P_1 P_2 P_4 P_5 \ldots .$

and the completed scale $\quad \left\{\begin{array}{l} P_1 P_2 P_3 P_4 P_5 P_6 \ldots . \\ P'_1 P'_2 P'_3 P'_4 P'_5 P'_6 \ldots . \end{array}\right.$

are each of them periodic, the period of the indices being $3i$. We may, accordingly, describe pertactile by the simpler name of periodic points. Every complete set of periodic points forms a closed system. By a complete set is to be understood the $9i^2$ subderivatives of the 9 points of inflexion, and by a closed system is to be understood one such that every connective and tangential of the points which it contains is itself a point of the system. According to what law such closed system may be resolved into partial closed systems must form the subject of further inquiry. When $i = 2$, the complete closed system of 36 points we know is resoluble into nine closed systems, each containing one point of inflexion and its three collinear anti-tangentials, and also, in four different ways, into three closed systems, each containing a collinear set of inflexions and their three sets of anti-tangentials.

We are now in a position to solve the problem of in-and-exscribed k-laterals.

Suppose $k = 3$, then $2^3 + 1 = 3i$ where $i = 3$, and the point P_1 will coincide with the point P_8, provided P_3 is a point of inflexion. So that the apices of the in-and-exscribed triangles are the 81 points which satisfy the equation $P_3 = P'_6$, of which 9 will correspond to the points of inflexion and 72 remaining over will give 24 finite triangles. If we denote by p, p', p'' three consecutive points in a straight line at any point of inflexion, $pp', p'p'', p''p$ form an infinitesimal triangle degenerating into a straight line, and this furnishes an improper solution of the question.

Calling $M, N, xyz\Omega$ the co-ordinates of P_3 when $P_1 = x, y, z$, the 72 points are given by combining the equation $MN\Omega = 0$ with the equation to the curve.

If $k = 4$, we make $2^4 - 1 = 3i$ where $i = 5$, and if $P_1 = P_{3i-1}$, we have also $P_1 = P_{3i+1}$; and the apices of the quadrilateral are found by making P_i, i. e. P_5, a point of inflexion.

The general form of P_5 being xU, yV, zW, the proper subderivatives P_5 result from $UVW = 0$ combined with the equation to the cubic, and there result $\frac{9(25-1)}{4}$, i. e. 54 in-and-exscribed quadrilaterals.

Each point of inflexion may still be regarded as yielding an improper solution of the question, since pp', $p'p''$, $p''p'$, $p'p$ may be viewed as a degenerate infinitesimal quadrilateral.

So when $k = 5$, making $2^5 + 1 = 3i$, $i = 11$; and there will result $\frac{9\,(11^2 - 1)}{5} = 216$ in-and-exscribed pentagons.

Likewise, since $\frac{2^7 + 1}{3} = 43$, there result $9\,\frac{43^2 - 1}{7}$, i. e. 9.264 or 2376 in-and-exscribed heptagons.

Let us now consider a case of k a composite number, and to fix the ideas, suppose $k = 15$. Make $\frac{2^{15} + 1}{3} = i$, then $i = 10923$. $\frac{2^{15} + 1}{3}$, *by virtue of its form*, contains the factors $\frac{2^3 + 1}{3}$ and $\frac{2^5 + 1}{3}$, i. e. 3 and 11, and is in fact equal to $3 \,.\, 11 \,.\, 331$. P_i will therefore be of the form $xU_3U_{11}\mathbf{U}$, $yV_3V_{11}\mathbf{V}$, $zW_3W_{11}\mathbf{W}$ (xU_3, yV_3, zW_3 corresponding to P_3, and xU_{11}, yV_{11}, zW_{11} to P_{11}).

Accordingly $\mathbf{U}$, $\mathbf{V}$, $\mathbf{W}$ will each be of the degree $(3 \,.\, 11 \,.\, 331)^2 - 3^2 - 11^2 + 1$, and the equation $\mathbf{UVW} = 0$, combined with the equation to the curve, will give the apices of the in-and-exscribed quindecagons, not including the improper solutions due to the points of inflexion, nor those due to the apices of the in-and-exscribed triangles or pentagons, which, in a certain but improper sense, each belong to the case of quindecagons. The number of apices of the proper quindecagons will therefore be $9\,[(3 \,.\, 11 \,.\, 331)^2 - 3^2 - 11^2 + 1]$, comprising sub-inflexions of several grades, as follows: $9\,(331^2 - 1)$ of the 331th grade, $9\,(3^2 - 1)\,(11^2 - 1)$ of the 33d grade, $9\,(3^2 - 1)\,(331^2 - 1)$ of the 993d grade, $9\,(11^2 - 1)\,(331^2 - 1)$ of the 3641th grade, and $9\,(3^2 - 1)\,(11^2 - 1)\,(331^2 - 1)$ of the 10923d grade.*. The above number of apices may be written $9\,[11^2 3^2\,(331^2 - 1) + (3^2 - 1)\,(11^2 - 1)]$, so that the number of quindecagons is $9\,[11^2 \,.\, 3^2 \,.\, 22 \,.\, 33 + 8^2]$.

It may be noticed that the primitive algebraical factor of $2^{15} + 1$, viz. 331, is a prime number. But the primitive part of $2^k - 1$ (k being even) or $2^k + 1$ (k being odd), i. e. $2^k - 1$ or $2^k + 1$ stripped of its obligatory factors dependent algebraically on the prime factors of k, may be a composite number.

Thus, let us suppose $k = 9$, the problem being that of finding the nature and number of the in-and-exscribed nonagons. Here $i = \frac{2^9 + 1}{3} = 171$, $2^9 + 1$ having, besides the obligatory factor $2^3 + 1$ due to its algebraical form, the *two* factors 3 and 19.

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* It is obvious that any derivative of an inflexion is itself an inflexion. For instance, if J is an inflexion, J_2 is the same as J, and J_3 (viz. J', J_2) is either J, J_2 i. e. J, or $(I, J), J_2$ i. e. $(I, J), J$ i. e. I (I being some other point of inflexion). Hence if $P_{\frac{i}{\delta}}$ is an inflexion, P_i is also an inflexion.

Taking each divisor of 171, viz. $3, 9, 19, 57, 171$, we see that the 3d, 9th, 19th, 57th, and 171th subderivatives of the nine points of inflexion will each of them be an apex of an in-and-exscribed nonagon. Of these, the 3d subderivatives, and they only, give improper solutions of the problem, they being the apices of the in-and-exscribed triangles. Hence the aggregate of proper apices and the corresponding nonagons separate into four distinct groups, corresponding to the primitive subderivatives of the 9th, 19th, 57th, and 171th grades respectively, of the inflexions. The number of the nonagons belonging to the several groups will be the quadritotients of $9, 19, 57, 171$, i. e. $9^2 - 9, 19^2 - 1$, $(19^2 - 1)(3^2 - 1), (9^2 - 9)(19^2 - 1)$ respectively, i.e. $171^2 - 9$, exactly the same as if 57 had been a prime number N, in which case the $(3N)^2$ subderivatives of an inflexion of the grade $3N$ would be subject to the deduction of $9 - 1$ for in-and-exscribed triangles, and 1 for the point itself.

To make more clear the distinct solutions of which the problem of in-and-exscription of a k-lateral in general admits, consider the case of $k = 8$. Here

$$i = \frac{2^8 - 1}{3} = \frac{2^4 - 1}{3}(2^4 + 1) = 85.$$

The first factor (the one algebraically contained in i) is 5 and the primitive algebraical factor is 17. The total number of octagonal apices will be $9(85^2 - 5^2)$, the number 5^2 corresponding to the points of inflexion and the in-and-exscribed quadrilaterals. These $255^2 - 15^2$ apices will consist of points of the form $I_{\frac{1}{17}}$ and $I_{\frac{1}{85}}$, the number of the former being $9(17^2 - 1)$ and of the latter $9(17^2 - 1)$ $(15^2 - 1)$.

It is easily seen that, in general, the number of apices of in-and-exscribed k-laterals is nine times the *functional totient* of $\left(\frac{2^k - \bar{1}^k}{3}\right)^2$, or, what is the same thing the number of apices is the functional totient of $(2^k - \bar{1}^k)^2$, as previously stated in Note to Proem in the last number of the *Journal;* the number of k-laterals is, of course, the number of apices divided by k. For instance, we thus have for the number of apices of quindecagons, nonagons, and octagons, respectively,

$$(2^{15} + 1)^2 - (2^3 + 1)^2 - (2^5 + 1)^2 + (2^1 + 1)^2,$$
$$(2^9 + 1)^2 - (2^3 + 1)^2, \quad (2^8 - 1)^2 - (2^4 - 1)^2,$$

as found above.

Since i is odd, every divisor of i will necessarily be so too. Conversely, it is easy to prove that every odd subderivative of a point of inflexion is an apex of an in-and-exscribed polygon, and to determine the number of its sides. For let i, any odd number, be given, and let k be the least number which will satisfy the condition that $2^k - \bar{1}^k$ shall be a multiple of $3i$, then the sub-inflexions of the i^{th} grade will be the apices of an in-and-exscribed k-lateral. I give, in the annexed table, the values of k corresponding to a given value of i, which, of course, are unique; whereas to a given value of k, in general, several values of i will correspond.

i	3	5	7	9	11	13	15	17	19	21	23	25	27
k	3	4	6	9	5	12	12	8	9	6	22	20	27

to which may be subjoined the reciprocal table

$k = 3$ $\quad i = 3$
$k = 4$ $\quad i = 5$
$k = 5$ $\quad i = 11$
$k = 6$ $\quad i = 7, 21$
$k = 7$ $\quad i = 43$
$k = 8$ $\quad i = 17, 85$
$k = 9$ $\quad i = 9, 19, 57, 171$
$k = 10$ $\quad i = 31, 341$
$k = 11$ $\quad i = 683$
$k = 12$ $\quad i = 13, 15, 35, 39, 65, 91, 195, 273, 455, 1365.$

To illustrate the way in which this table is formed, take the case of $k = 12$; then $\frac{2^{12} - 1}{3} = 3 . 5 . 7 \times 13$ where 3 belongs to $k = 3$, 5 to $k = 4$, 7 to $k = 6$; the values of i are found by taking the divisors of 1365, except those which are found set against $k = 3$, $k = 4$, $k = 6$, i. e. 3, 5, 7, 21.

The successive tangentials of any even-graded inflexional subderivative as $2^q i$, where i is odd, will evidently consist of a chain of q points attached to the ring formed by the apices of an in-and-exscribed polygon of k sides, where k is the least number which makes $2^k \pm 1$ divisible by $3i$.

In all cases (since k is to have the minimum value which makes $\frac{2^k \pm 1}{3}$ contain i) $2k$ must be $\tau(3i)$ or a submultiple of it, so that, if $i = 3^q j$, k is either $3^q \tau j$ or a submultiple of it; when $i = 3^q$, since the cyclotomic functions of the first species $\chi_3 2$, $\chi_{3^2} 2$, ... $\chi_{3^q} 2$ can only contain the first power of the intrinsic divisor 3, it follows that $k = 3^q = i$, as is seen in the table to be the case for $i = 3, 9, 27$; or, in other words, a 3^qth subderivative of a point of inflexion is an apex of an in-and-exscribed polygon of 3^q sides.

It may be as well to mention again here, by way of a remind, that the number of in-and-exscribed k-laterals whose apices are i^{th} subderivatives of the inflexions, is always the k^{th} part of nine times the quadritotient of i; when $i = 3^q$ this number will be $\frac{1}{3^{q-2}}\{3^{2q} - 3^{2q-2}\}$, i. e. $3^{q+2} - 3^q$, being thus $24, 72$, 216, etc., for triangles, nonagons, eikosiheptagons, etc.

TITLE 5. — *An Exact Proof of the Scalar Law of Squares.*

I will now give an exact proof of the law that the order in the variables of P_n is n^2 in regard to the co-ordinates of P, and furthermore that the co-ordinates when $i = 3m \pm 1$ are of the form xU, yV, zW, and when $i = 3m$ are of the form $M, N, xyz\Omega$; x, y, z being the co-ordinates of the primitive P_1 and U, V, W, M, N, Ω quantics in x^3, y^3, z^3. Of course the order of a point means the order of its system of co-ordinates *expressed in its lowest terms*, that is to say when the values of the three co-ordinates have no common measure, and consequently the co-ordinates of any *two* of them are relatively prime in an algebraical sense, as follows from the equation $X^3 + Y^3 + Z^3 + kXYZ = 0$.

The law to be established comprises, it will be seen, two elements,— one numerical, the *rule of squares*; the other formal, containing two rules, one regarding the *distribution* of x, y, z between the co-ordinates, the other the quanticity of the parts not multiplied by x, y, z or xyz in respect to x^3, y^3, z^3.

Let us suppose that the law is true up to n inclusive. I shall show that it is true up to $2n$ inclusive.

1° For the case of $2i$ where $i \overset{=}{<} n$.

Let X, Y, Z be the system of co-ordinates to P_i in its lowest terms; then, by the law of compound derivation, P_{2i} is $X(Y^3 - Z^3), Y(Z^3 - X^3), Z(X^3 - Y^3)$.

If these regarded as functions of X, Y, Z had any common measure X, Y or $X, Z^3 - X^3$ would have a common measure. Hence X, Y, Z would all have a common measure. Nor can they have any common factor F, a function of x, y, z. For in that case, when $F = 0$, we should have

$$Y^3 - Z^3 = 0 \text{ , } Z^3 - X^3 = 0 \text{ or } X^3 = Y^3 = Z^3,$$

and the arbitrary parameter k would be $-3 \,.\, 1^{\frac{1}{3}}$, so that the cubic would become a triplet of straight lines, a supposition which falls outside the pale of the question.

Hence P_{2i} will be of four times the order of P_i, and therefore, by hypothesis, of the order $4i^2$, i. e. $(2i)^2$. Also, obviously, the form xU, yV, zW or $M, N, xyz\Omega$ (as the case may be) which exists for i is maintained for $2i$, which is or is not divisible by 3 according as i is or is not so divisible.

2° Let the index be any odd number less than $2n$.

I shall first establish a Lemma concerning the co-ordinates given by my formulæ for the connectives of P, Q and P', Q, where P' is the opposite to P in respect to a given point of inflexion (say $x = 1, y = -1$), and $x^3 + y^3 + z^3 + kxyz = 0$ is the equation to the cubic.

The connectives of $\begin{matrix}(u, v, w)\\(u', v', w')\end{matrix}$ and of $\begin{matrix}(v, u, w)\\(u', v', w')\end{matrix}$

are represented respectively by

$$\left.\begin{matrix} vwu'^2 - v'w'u^2 \\ wuv'^2 - w'u'v^2 \\ uvw'^2 - u'v'w^2 \end{matrix}\right\} \text{ and } \left\{\begin{matrix} uwu'^2 - v'w'v^2 \\ wvv'^2 - w'u'u^2 \\ vuw'^2 - u'v'w^2, \end{matrix}\right.$$

the 3^d co-ordinate being the same in both systems, which, of course, remain to be reduced to their simplest terms, being at present each of the order $2i^2 + 2j^2$.

I say that the same quantity F cannot divide each of the two sets of quantities when u, v, w; u', v', w' are derivatives, one of an even, the other of an odd grade of the same point on the cubic.

For, if so, let $F = 0$; then each quantity in the two systems becomes zero.

Call $\frac{u}{w}, \frac{v}{w}; \frac{u'}{w'}, \frac{v'}{w'},$ $r, s; r', s'$ respectively.

Then

$$\begin{matrix} (1) \ldots . sr'^2 - s'r^2 = 0 & rr'^2 - s's^2 = 0 \ldots . (3) \\ (2) \ldots . rs'^2 - r's^2 = 0 & r'r^2 - ss'^2 = 0 \ldots . (4) \end{matrix}$$

$$(5) \ldots . rs = r's'.$$

Writing $r^3 = R$, $s^3 = S$, $r'^3 = R'$, $s'^3 = S'$; 5, (3, 4), (1, 2) respectively give $RS = R'S'$, $RR' = SS'$, $R'S = RS'$. The second and third of these combined give $R^2 = S^2$, $R'^2 = S'^2$ and the first and second combined give $R'^2 = S^2$. Hence, $R^2 = R'^2 = S^2 = S'^2$, and consequently the original equations (1), (2), (3) give $S = S'$, $R = R'$, $R = S'$ or $r^3 = s^3 = r'^3 = s'^3$.

Let $r = \alpha s$, $r' = \beta s'$, $s = \gamma s'$. Then $\alpha^3 = \beta^3 = \gamma^3 = 1$, and all the equations (1), (2), (3), (4), (5) will easily be found to be satisfied when (and only when) $\alpha = \beta\gamma$.

The equations $r^3 = s^3$, $r'^3 = s'^3$, i. e. $u^3 = v^3$, $u'^3 = v'^3$, imply that the points P, Q are two either distinct or identical anti-tangentials to the same point of inflexion $x = 1, y = -1$. I say that this is impossible when P, Q are derivatives of the degrees i, j of the same point U on the curve, if $i + j$ is an odd number. It must be noticed that P and Q (two Plückerian points belonging to the same point of inflexion I) are identical with P' and Q' respectively.

Any even-degreed derivative of P or Q is I, and any odd-degreed derivative is the same point P or Q over again.

Let now $i\mu - j\nu = 1$. Then $U = U_{i\mu - j\nu}$ will be (without regard to the modulus 3) the connective of $U_{i\mu}$ and $U_{j\nu}$, because we may substitute at will U_i' for U_i and U_j' for U_j. But $U_{i\mu}$ and $U_{j\nu}$, if μ, ν be both odd, will be U_i and U_j over again, or if μ, ν be one odd and the other even, will be I and one of the two Plückerian points.

Hence U is the connective of I and a Plückerian point, or else of two Plückerians which are identical, or of two Plückerians (both appurtenant to I) which are distinct.

In the 1st and 3d cases, then, U is a Plückerian, in the 2d case a point of inflexion. But every derivative of a point of inflexion is a point of inflexion, and every even-degreed derivative of a Pluckerian is also a point of inflexion; but by hypothesis (since one of the two numbers i, j is even) an even-degreed derivative of U is a Plückerian, which is self-contradictory. Hence, it follows that the expressions given by *my* formulæ for the connectives of P_i, P_j and P_i', P_j when $i + j$ is odd, say P, Q, R; P', Q', R, cannot have a common factor; so that if M is a common measure of P, Q, R and M' of P', Q', R', M is relatively prime to M'.

Let ϕ, ψ, ω be always understood to mean $\phi(x^3, y^3, z^3)$, $\psi(x^3, y^3, z^3)$, $\omega(x^3, y^3, z^3)$; let $(\mu), (\nu)$ be understood to mean the prime systems of co-ordinates u, v, w; u', v', w' which represent μ, ν (μ and ν being numbers, accented or unaccented, representing derivatives to the index μ and ν) let $[\mu, \nu]$ represent the unreduced system of the co-ordinates of the connective of μ, ν, viz. $v'w'u^2 - vwu'^2$, $w'u'v^2 - wuv'^2$, $u'v'w^2 - uvw'^2$; (μ, ν) the above system *reduced* by elimination of the greatest common measure of its terms.

If $(\mu), (\nu)$ are each of the form $x\phi, y\psi, z\omega$, $[\mu, \nu]$ is of the form $x^2yz\phi_1$, $xy^2z\psi_1$, $xyz^2\omega_1$, but $[\mu', \nu]$, i. e. the unreduced connective of $y\psi, x\phi, z\omega$; $x\phi', y\psi', z\omega'$, is of the form $z\phi_1, z\psi_1, xyz^2\omega_1$.

Again, if (μ) is of the form $x\phi, y\psi, z\omega$ and (ν) of the form $\phi_1, \psi_1, xyz\omega_1$, $[\mu', \nu]$, the unreduced connective of the systems $y\psi, x\phi, z\omega$ and $\phi_1, \psi_1, xyz\omega_1$, is easily seen to be of the form $zx\Phi, zy\Psi, z^2\Omega$.

Furthermore, the order in the variables of (p') is obviously the same as that of (p).

Now it has been shown under Title 2 that

$$6i - 1 = \overline{3i-1}', 3i \qquad 6i - 5 = \overline{3i-3}', \overline{3i-2}' \qquad 6i - 3 = \overline{3i-2}', 3i - 1.$$

If, then, $(3i)$ and $(3i - 3)$* are of the form $\phi, \psi, xyz\omega$, and $(3i - 2), (3i - 1)$

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* $\overline{3i-3}'$ will obviously be of the same form as $3i - 3$.

each of the form $x\phi, y\psi, z\omega$, it follows that $[6i - 1]$ and $[6i - 5]$ will be of the form $zx\phi, zy\psi, z^2\omega$ and $[6i - 3]$ of the form $z\phi, z\psi, xyz^2\omega$.

The above inference suffices to show that, if, for all values of $3\mu \pm 1$ and 3μ up to n inclusive, it be true that $(3\mu \pm 1)$ is of the form $x\phi, y\psi, z\omega$ and of the order $(3\mu \pm 1)^2$, and (3μ) is of the form $\phi, \psi, xyz\omega$ and of the order $(3\mu)^2$; then the same will be true up to $2n$ inclusive.

That this is true for even values not exceeding $2n$ appears from what has been already shown. Confining, then, our attention to odd numbers less than $2n$; these must be representable by $6i - 5, 6i - 3$ or $6i - 1$, and by hypothesis the form of each of the systems $(3i), (3i - 1), (3i - 2), (3i - 3)$ fulfils the conditions of the last paragraph but one; consequently the form of $[6i - 5]$, $[6i - 3]$, $[6i - 1]$ will be $zx\phi, zy\psi, z^2\omega$; $z\phi, z\psi, xyz^2\omega$; $zx\phi, zy\psi, z^2\omega$, i. e. in every case the factor z will be contained in each term of the system $[\overline{\iota - 1}', \iota^?]$, which represents an unreduced system of co-ordinates of the point $2\iota - 1$, the mark of interrogation signifying a blank or an accent as the case may be.

But either the point 1 or the point 1′ will, in every case, correspond to the connective obtained by changing $\overline{\iota - 1}'$ into $\iota - 1$;* moreover, the unreduced system of co-ordinates to that connective will have the third term, say π, in common with the unreduced system to $2\iota - 1$ above mentioned.

This contrary system we know must have the common factor $\frac{\pi}{z}$ because 1 and 1′ are denoted by x, y, z; y, x, z respectively. Hence the unreduced system for $2\iota - 1$ can have no other common factor except z, which they have been shown to have; since, were it otherwise, the *two* contrary systems would have some quantity contained in $\frac{\pi}{z}$ for a joint common measure, which has been proved to be impossible.

Hence, the form of $(2\iota - 1)$ is $x\phi, y\psi, z\omega$ or $\phi, \psi, xyz\omega$ according as $2\iota - 1$ is not or is divisible by 3, and its order is in all cases $2\overline{\iota - 1}^2 + 2\iota^2 - 1$, i. e. $(2\iota - 1)^2$.

Hence the form-law of distribution of the simple powers of the variables x, y, z and of the quanticity in x^3, y^3, z^3 of the multipliers of x, y, z or of $1, 1, xyz$, as well as the numerical law that the order of any derivative is the square of its index, will be true up to $2n$ inclusive if true up to n inclusive; and being true for $n = 1$, is true universally.

As a corollary we may now do away with the restriction of $i + j$ being odd, and affirm that in all cases (the futile one of $i = j$ alone excepted), if the reduced system of co-ordinates to the connective of P_i, P_j be F, G, H and to that of P_i', P_j be F', G', H', then the unreduced system expressing those connectives given by my formulæ of connection will be $H'F, H'G, H'H$; HF', HG', HH', respectively; for the two systems of unreduced co-ordinates (each of the order $2i^2+2j^2$) contain, one of them a common factor of the order $(2i^2+2j^2)-(i-j)^2$ i. e. $(i+j)^2$, the other a common factor of the order $(2i^2+2j^2)-(i+j)^2$ i. e. $(i-j)^2$, and these two factors being prime to each other, their product must be contained in the term common to the two systems, and being of the same order $(i+j)^2+(i-j)^2$ as that common term, must be equal to it.

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* For, on consulting Title 2, it will be found that *in every case*, if the arithmetical value of the index of P_i, P_j is $i \pm j$, that of P_i', P_j is $(\iota \mp j)^?$.

Hence, if π be the common unreduced term, and H, H' the two reduced terms, we must have $\pi = \frac{\pi}{H}\cdot\frac{\pi}{H'}$ or $\pi = HH'$, as was to be shown.

As a matter rather of curiosity than of real importance I will state the analogous law when the connective and cross-connective between two derivatives is expressed by Cauchy's formulæ instead of my own. These formulæ, it will be remembered, give for the co-ordinates of the connective of u, v, w; u_1, v_1, w_1 the minor determinants of the matrix

$$\begin{vmatrix} vw_1 - v_1w \; ; & w_1u - wu_1 \; ; & uv_1 - u_1v \\ uu_1 \; ; & vv_1 \; ; & ww_1 \end{vmatrix}$$

If, now, the prime system of co-ordinates to the connectives of P_i, P_j; P_i', P_j be denoted as before by F, G, H; F', G', H', I find by calculation that the Cauchian formulæ will present these two systems under the unreduced forms

$$\begin{array}{l} (F'+G')F, \; (F'+G')G, \; (F'+G')H \\ (F+G)F', \; (F+G)G', \; (F+G)H', \end{array}$$

between which there is no common term; and consequently, had I not discovered my own simpler formulæ, the method of proof of the Law of Squares which I have employed would have been inapplicable, and it is not easy to see what other strict method of proof could have taken its place.

I have thus accomplished the very difficult task of proving a negative, in this instance the non-existence of *latent* common factors to the co-ordinates of the connective of any two given derivatives. I might have founded a much easier proof of the Law of Squares upon Mr. Franklin's geometrical solution of the problem of finding the number of in-and-exscribed k-laterals to a cubic (if one could feel quite assured *à priori* of the strict logic of the process*) as follows: He has virtually found (*vide* last number of the Journal) that the number of apices of the in-and-exscribed k-laterals of *every kind* [and not excluding the points of inflexion] is $(2^k - 1^k)^2$. If, then, $2^k - \bar{1}^k = 3\,i$, it follows from what has been shown in the preceding pages, that the order of P_i in the co-ordinates of P is $\frac{1}{9}\,(3\,i)^2$, i. e. i^2.

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* In that solution the apices are found as the intersections of the cubic with another curve. Certain of these intersections are seen from geometrical considerations to count twice, and others three times; but while we have no reason to suppose any further cause of reduction, the non-existence of such cause is not proved.— F. F.

Let now i' be any number whatever, and τ the totient of $3\,i'$; then τ is even, and, by Fermat's Theorem, $2^\tau - \bar{1}^\tau = 3\,i'i''$.

Hence, if μ', μ'' are the orders of $P_{i'}$, $P_{i''}$ respectively, the law of compound derivation will suffice to lead to the conclusion that $\mu'\mu''$ will be the order of $P_{i'i''}$, and accordingly $\mu'\mu'' = i'^2 i''^2$; but $\frac{\mu'}{i'^2}$, $\frac{\mu''}{i''^2}$, it has been proved under a preceding Title, are neither of them greater than unity: hence each of them is equal to unity, and i'^2 is the order of $P_{i'}$, as was to be shown.

ADDENDUM ON THE DEGORDER OF THE DERIVATIVES TO A POINT ON A CUBIC IN THE NATURAL SCALE.

Let n be any number not divisible by 3. The n^{th} derivative, it has been proved, is of the order n^2 in the variables. It remains to determine its *degree in the coefficients.*

When $n = 2$ we know that the degorder is $[4\,;4]$, each new co-ordinate being one of the minors of the rectangular matrix

$$\left|\begin{array}{ccc} \frac{dU}{dx} & \frac{dU}{dy} & \frac{dU}{dz} \\ \frac{dH}{dx} & \frac{dH}{dy} & \frac{dH}{dz} \end{array}\right|$$

where U is the cubic and H its Hessian.

Suppose ν to be the degree in the coefficients of the n^{th} derivative. Then the degree of the $(2\,n)^{\text{th}}$ derivative regarded as the second of the n^{th} will be $4\,\nu + 4$, and regarded as the n^{th} of the second will be $n^2 . 4 + \nu$, and these two must be equal. Hence $3\,\nu = (n^2 - 1)\,4$ or $\nu = \frac{4}{3}\,(n^2 - 1)$.

Hence the degorder of any n^{th} derivative in the natural scale is $\left[\frac{4\,n^2 - 4}{3}\,;\; n^2\right]$. If we substitute the co-ordinates of this derivative in the given cubic U, the result must be of the form $U . R$ and will be of the degorder $[1 + 4\,n^2 - 4\,;\, 3\,n^2]$. Hence R is of the degorder $[4\,n^2 - 4\,;\; 3\,n^2 - 3]$. If the well-known covariant of the degorder $[12\,; 9]$ be called J, R is of the same degorder as $J^{\frac{n^2-1}{3}}$, and possibly may be found to be identical with it. To corroborate the validity of the determination of the degorder of the n^{th} derivative, we may proceed as follows:

Imagine, at first, the cubic to be reduced to the canonical form $x^3 + y^3 + z^3 - 3\,kxyz$. The connective of P_1, P_2 in its reduced form is x, y, z; but in its unreduced form and prior to all simplification, will, by virtue of the theory (Titles 1 and 5), be of the form Mx, My, Mz where

$$M = x^3y^6 + y^3z^6 + z^3x^6 + x^6y^3 + y^6z^3 + z^6x^3 - 6\,x^3y^3z^3$$
$$+ kxyz\,(x^6 + y^6 + z^6 - y^3z^3 - z^3x^3 - x^3y^3);\,*$$

consequently M expressed (as I shall hereafter suppose) in terms of the original coefficients and variables, will be of the degorder $[9\,;\,9]$: for Mx, My, Mz are of the degorder $[1 + 2\,.\,4\,;\,2\,(1 + 4)]$, i. e. $[9\,;\,10]$.† Also the degorder of P_4 will be $[4 + 4\,.\,4\,;\,16]$, i. e. $[20\,;\,16]$.

Suppose now we wish to find the degorder of P_5.

The unreduced connective of P_1, P_4 will be of the form MX, MY, MZ, where X, Y, Z are the reduced co-ordinates and M is exactly the same thing as before. The degorder of the unreduced co-ordinates will be $[1 + 2\,.\,20\,;\,2\,(1 + 16)]$, i. e. $[41\,;\,34]$; and consequently, subtracting $[9\,;\,9]$, the degorder of X, Y, Z will be $[32\,;\,25]$, i. e. $\left[4\,\frac{5^2-1}{3}\,;\,5^2\right]$.

So, again, to find P_7 we may regard it as the connective of P_2, P_5. The unreduced degorder of P_7 will thus be seen to be $[1 + 2\,(4 + 32)\,;\,2\,(4 + 25)]$, i. e. $[73\,;\,58]$, and subtracting, as before, $[9\,;\,9]$, the degorder of the reduced co-ordinates of P_7 becomes $[64\,;\,49]$ i. e. $\left[4\,\frac{7^2-1}{3}\,;\,7^2\right]$, agreeable to what has been previously found; and so, in general, supposing the degrees of P_μ and $P_{\mu+3}$ in the coefficients to be $4\,\frac{\mu^2-1}{3}$ and $4\,\frac{(\mu+3)^2-1}{3}$, the unreduced degree of $P_{2\mu+3}$ will be $1 + 8\left\{\frac{\mu^2-1}{3} + \frac{(\mu+3)^2-1}{3}\right\}$, from which subtracting 9, the reduced degree becomes $8\left\{\frac{2\,\mu^2 + 6\,\mu + 4}{3}\right\}$, which is the same thing as $4\left\{\frac{(2\,\mu + 3)^2 - 1}{3}\right\}$, as ought to be the case. There is, therefore, no loophole for doubt left open as regards the degorder of any *natural* derivative to the index k (a number necessarily of the form $3\,i \pm 1$) being $[\frac{4}{3}\,(k^2 - 1)\,;\,k^2]$, a notable result!

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* It is worthy of remark that, if we make $U = 0$, so that $3\,kxyz$ becomes equal to $x^3 + y^3 + z^3$, the expression in the text for M gives $3\,M$ equal to the norm of $x + 1^{\frac{1}{3}}y + 1^{\frac{1}{3}}z$, namely, $(x^3 + y^3 + z^3)^3 - 27\,x^3y^3z^3$.

† In fact, M, as may easily be shown, is the covariant $\left[\Sigma\left(\frac{dU}{dy}\cdot\frac{dH}{dz} - \frac{dU}{dz}\cdot\frac{dH}{dy}\right)\frac{d}{dx}\right]^2 U$, in other words the symmetrical determinant of the 5th order formed by double-bordering the Hessian matrix with the differential derivatives of the Hessian and of the original cubic.

We are now in possession of a method for finding any natural derivative to the index n. If n is even, it may be derived immediately from the derivative to the index $\frac{n}{2}$. If n is odd, it must be of the form $2\mu+3$ where μ is not divisible by 3.

Taking P as the initial point, P_μ and $P_{\mu+3}$ may be considered as known. Calling their co-ordinates X, Y, Z; X_1, Y_1, Z_1 respectively, and substituting $\lambda X+\mu X_1$, $\lambda Y+\mu Y_1$, $\lambda Z+\mu Z_1$ in the equation to the cubic, we shall obtain an equation of the form $\lambda^2\mu B+\lambda\mu^2 C=0$. The unreduced co-ordinates of $P_{2\mu+3}$ will then be $CX-BX_1$, $CY-BY_1$, $CZ-BZ_1$, which will contain a common measure M of the degorder [9; 9], and $\frac{CX-BX_1}{M}$, $\frac{CY-BY_1}{M}$, $\frac{CZ-BZ_1}{M}$ will be the expression for the point $P_{2\mu+3}$ in its simplest terms.

More generally, if $n=2\mu+3i$, we may obtain, in like manner as above, the unreduced co-ordinates of the connective to P_μ, $P_{\mu+3i}$, and, by an easy calculation, it will be found that the new common measure will be of the degorder $[12i^2-3; 9i^2]$, and will be constant, i. e. independent of μ for any given value of i, and identical with the common measure to the unreduced co-ordinates of P_{3i+2} regarded as the connective to P and P_{3i+1}.

It is well worthy of remark that if X, Y, Z be the co-ordinates of any derivative, and ξ, η, ζ contragredient to x, y, z, $X\xi+Y\eta+Z\zeta$ will be an invariantive concomitant to the given cubic. This gives rise to a new series of reflexions, the development of which must be deferred to a more convenient occasion.*

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* It is obviously a step towards the attainment of the desideratum of finding the general expression for any derivative in an explicit form, or, at all events, by explicit processes and without the necessity for division of the unreduced co-ordinates by a common measure. This latter, it should be observed however, by virtue of what is stated above, is always known *à priori.*